Noncommutative surfaces, clusters and their symmetries

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Plan

1. Non-commutative cluster algebras–an informal introduction
2. Non-commutative polygon
3. Non-commutative surfaces and their cluster structures
4. Braid group action and their symmetries

Noncommutative clusters, informal introduction

A (noncommutative) cluster structure on a graded $\mathbb{K}$-algebra $\mathcal{A}$ consists of a certain graded group $Br_{\mathcal{A}}$ together with a collection of homogeneous embeddings ι of a given graded group G into the multiplicative monoid $\mathcal{A}^{\times}$ (these embeddings are referred to as noncommutative clusters) and a faithful homogeneous action $\triangleright_{\iota}$ of $Br_{\mathcal{A}}$ on G for any ι such that:

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- The extensions $\iota : \mathbb{K}G \to \mathcal{A}$ are injective and their images generate $\mathcal{A}$ (and $\mathcal{A}$ is a isomorphic to a noncommutative localization of $\mathbb{K}G$).
- (monomial mutation) For any ι and ι' we expect a (unique) automorphism $\mu_{\iota,\iota'}$ of G which intertwines between ι and ι' as well as between $Br_{\mathcal{A}}$-actions $\triangleright_\iota$ and $\triangleright_{\iota'}$.

Noncommutative clusters, informal introduction

- For any cluster homomorphism $f:\mathcal{A}\to\mathcal{A}'$ we expect a unique (up to conjugation) group homomorphism $f_*:G\to G'$ so that the induced homomorphism $Br^f_{\mathcal{A}}:=\{T\in Br_{\mathcal{A}}:T(Ker\ f_*)=Ker\ f_*\}\to Br_{\mathcal{A}'}$ is injective.

  In many cases we expect a (noncommutative) Laurent Phenomenon:

- Given a cluster $\iota:G\hookrightarrow\mathcal{A}^\times$, for any cluster $\iota':G\hookrightarrow\mathcal{A}^\times$ there is a submonoid $M_{\iota'}\subset G$ generating G such that $\iota'(M_{\iota'})$ is in the semiring $\mathbb{Z}_{\geq 0}\iota(G)$, moreover,

$$\iota'(m)=\iota(\mu_{\iota,\iota'}(m))+\text{lower (upper) terms in }\iota(G)$$

  for any $m\in M_{\iota'}$.

Example: Ordinary and quantum cluster structures

The localization $\mathcal{A}$ of a (quantum) cluster algebra $\underline{\mathcal{A}}$, determined by an $m \times n$ exchange matrix $\tilde{B}$ (and compatible $m \times m$ skew-symmetric matrix Λ), by the set X of all of its cluster variables satisfies all of the above requirements with $G \cong \mathbb{Z}^m$ (or its central extension G_q in quantum case) so that $\mathbb{Q}G = \mathbb{Q}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ for a given cluster $\{x_1, \ldots, x_n\}$ in $\mathcal{A}$. The well-known commutative/quantum Laurent Phenomenon asserts that the set of all (quantum) cluster variables belongs to the group algebra $\mathbb{Q}G$ which is an instance of its noncommutative counterpart stated above. In these cases, $Br_{\mathcal{A}}$ is essentially the group of symplectic transvections introduced by B. Shapiro, M. Shapiro, A. Vainshtein, A. Zelevinsky in 2000) and it is always a quotient of an appropriate Artin braid group.

Non-commutative polygon

Non-commutative n-gon Σ_n

Σ_n: disk with n marked points, labeled clockwise by $1, 2, \cdots, n$, on the boundary;

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Definition [Berenstein-Retakh]

The *non-commutative* n-gon $\mathcal{A}(\Sigma_n)$ is defined to be the non-commutative algebra generated by $x_{ij}, i \neq j \in [n]$, subject to

- (triangle relation) $x_{ij}x_{kj}^{-1}x_{ki} = x_{ik}x_{jk}^{-1}x_{ji}$ for any triangle (i, j, k);
- (Ptolemy relation) $x_{jl} = x_{ji}x_{ki}^{-1}x_{kl} + x_{jk}x_{ik}^{-1}x_{il}$ for any quadrilateral (i, j, k, l) on Σ_n.

For any triangle (i, j, k), define $T_i^{jk} := x_{ji}^{-1} x_{jk} x_{ik}^{-1} = x_{ki}^{-1} x_{kj} x_{ij}^{-1}$.
Call it the *angle* at i.

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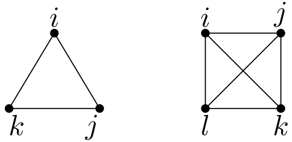
(Ptolemy relation) Angle is additive

$$x_{ji}^{-1}x_{jl}x_{il}^{-1} = x_{ki}^{-1}x_{kl}x_{il}^{-1} + x_{ji}^{-1}x_{jk}x_{ik}^{-1}, \qquad T_i^{jl} = T_i^{jk} + T_i^{kl}.$$



Non-commutative Laurent phenomenon

For any triangulation Δ of Σ_n, the total angle $T_i(\Delta)$ is defined to be

$$\sum_{(i,j,k)} T_i^{jk},$$

where (i,j,k) runs over all triangles incident to i in Δ.

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Proposition.

For any $i \in [n]$, the total angle $T_i(\Delta)$ does not depend on the triangulation, i.e., $T_i(\Delta) = x_{i^- i}^{-1} x_{i^- i^+} x_{i^+ i}^{-1} = T_i^{i^-, i^+}$.

Non-commutative Laurent phenomenon

Theorem [Berenstein-Retakh]

For any triangulation Δ of the n-gon and for any distinct $i, j \in [1, n]$, we have

$$x_{ij} = \sum_{\mathbf{i}=(i_1,\ldots,i_{2m})} x_{\mathbf{i}},$$

where the summation is over all $(ij)-$*admissible* sequences $\mathbf{i}$ in Δ, i.e.,

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- an edge (i_s, i_{s+1}) intersects (i, j) iff s is even;
- If $\mathbf{p} := (i_k, i_{k+1}) \cap (i, j) \neq \emptyset$ and $\mathbf{q} := (i_\ell, i_{\ell+1}) \cap (ij) \neq \emptyset$ for some $k < \ell$, then the point $\mathbf{p}$ of (ij) is closer to i than $\mathbf{q}$.

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and $x_{\mathbf{i}} := x_{i_1,i_2} x_{i_3,i_2}^{-1} x_{i_3,i_4} \cdots x_{i_{2m-1},i_{2m-2}}^{-1} x_{i_{2m-1},i_{2m}}$

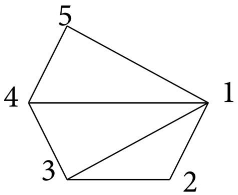
Non-commutative Laurent phenomenon

Example

If $n = 5$ and $\Delta = \{(1,3),(3,1),(1,4),(4,1);(i,i\pm 1)|i \in [5]\}$, then

$$x_{21}^{-1}x_{25}x_{15}^{-1} = x_{21}^{-1}x_{23}x_{13}^{-1} + x_{31}^{-1}x_{34}x_{14}^{-1} + x_{41}^{-1}x_{45}x_{15}^{-1}.$$

$$x_{25} = x_{23}x_{13}^{-1}x_{15} + x_{21}x_{31}^{-1}x_{34}x_{14}^{-1}x_{15} + x_{21}x_{41}^{-1}x_{45}.$$



Triangle group and braid group action

For any triangulation Δ of Σ_n, the *triangle group* $\mathbb{T}_\Delta$ is defined to be $\mathbb{T}_\Delta = \langle t_{ij}, (ij) \in \Delta \rangle$ subject to

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Proposition

For any two triangulations Δ, Δ' of Σ_n, $\mathbb{T}_\Delta \cong \mathbb{T}_{\Delta'}$.

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Proposition

For any two triangulations Δ, Δ' of Σ_n, $\mathbb{T}_\Delta \cong \mathbb{T}_{\Delta'}$.

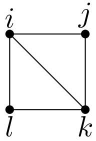
We denote $\mathbb{T}_{\Sigma_n} = \mathbb{T}_\Delta$ and call it *triangle group* of Σ_n.

Triangle group and braid group action

Fix a triangulation Δ of Σ_n, define $T_{ik} \in Aut(\mathbb{T}_\Delta)$

$$T_{ik}(t_\gamma) = \begin{cases} t_{ij}t_{kj}^{-1}t_{kl}t_{il}^{-1}t_\gamma & \text{if } \gamma = (ik) \\ t_\gamma t_{li}^{-1}t_{lk}t_{jk}^{-1}t_{ji} & \text{if } \gamma = (ki) \\ t_\gamma & \text{otherwise} \end{cases}$$

for any internal edge (ik) of Δ where $(ijkl)$ is a cyclic quadrilateral containing (ik) as a diagonal.



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For any two triangulations Δ, Δ' of Σ_n, $Br_\Delta \cong Br_{\Delta'} \cong Br_{n-2}$.

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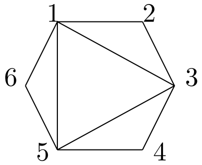
If Δ is a triangulation of the hexagon as in the picture



then $Br_{\mathcal{A}_6} = Br_4$ is generated by T_{13}, T_{15}, and T_{35} subject to

$T_{13}T_{35}T_{13} = T_{35}T_{13}T_{35}, T_{35}T_{15}T_{35} = T_{15}T_{35}T_{15}, T_{35}T_{15}T_{35} = T_{15}T_{35}T_{15}$

and $T_{13}T_{15}T_{35}T_{13} = T_{15}T_{35}T_{13}T_{15} = T_{35}T_{13}T_{15}T_{35}$.

Non-commutative marked surfaces and their cluster structure

Non-commutative marked orbifolds

$\Sigma = (S, M, U)$ marked orbifold, i.e.,

S Riemann Surface,

$M \subset S$ marked points, $|M| < \infty$

$U \subset S \setminus \partial S$ orbifold points, $|U| < \infty, M \cap U = \emptyset$

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$\Gamma(\Sigma)$: set of curves in Σ

Non-commutative marked orbifolds

Definition [Berenstein-H-Retakh]

The *non-commutative orbifold* $\mathcal{A}(\Sigma)$ is defined to be the non-commutative algebra generated by $x_\gamma, \gamma \in \Gamma(\Sigma)$, subject to

(1) (triangle relation) $x_{\gamma_1} x_{\bar{\gamma}_2}^{-1} x_{\gamma_3} = x_{\bar{\gamma}_3} x_{\gamma_2}^{-1} x_{\bar{\gamma}_1}$ for any triangle $(\gamma_1, \gamma_2, \gamma_3)$ in Σ;

(2) (Ptolemy relation) $x_{jl} = x_{ji} x_{ki}^{-1} x_{kl} + x_{jk} x_{ik}^{-1} x_{il}$ for any quadrilateral (i, j, k, l) in Σ;

(3) (Monogon relations) $x_{\overline{\gamma}} = x_\gamma$ for each loop γ cuts out a monogon which contains only an orbifold point.

Non-commutative orbifolds

Definition–continuous

(4) (Bigon orbifold relations)

$x_{\alpha'} = x_{\bar{\alpha}_1} x_\alpha^{-1} x_{\alpha_1} + 2\cos(\frac{\pi}{|p|}) x_{\bar{\alpha}_1} x_\alpha^{-1} x_{\bar{\alpha}_2} + x_{\alpha_2} x_\alpha^{-1} x_{\bar{\alpha}_2}$ for any bigon (α_1, α_2) around an orbifold point p, where α is the loop around p such that $(\alpha_1, \alpha_2, \alpha)$ is a triangle and α' is the loop around p such that $(\alpha', \alpha_2, \alpha_1)$ is a triangle.

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$^{-}: \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma, x_\gamma \mapsto x_{\bar{\gamma}}$ is an involution.

Remark

This is a non-commutative version of (generalized) cluster algebras from orbifolds.

Total angle

Let Δ be a triangulation of Σ.

- For any Δ and any cyclic triangle $(\gamma_1, \gamma_2, \gamma_3)$ in Δ denote

$$T_{\gamma_1,\gamma_2,\gamma_3} := x_{\gamma_3}^{-1} x_{\overline{\gamma}_2} x_{\gamma_1}^{-1} = x_{\overline{\gamma}_1}^{-1} x_{\gamma_2} x_{\overline{\gamma}_3}^{-1}.$$

- For any $i \in M$, denote

$$T_i(\Delta) = \sum T_{(\gamma_1,\gamma_2,\gamma_3)} + \sum 2cos(\frac{\pi}{|p|}) x_{\ell_p}^{-1},$$

where the first summation is over all clockwise triangles $(\gamma_1, \gamma_2, \gamma_3)$ in Δ such that $s(\gamma_1) = i$ and $T_{(\gamma_1,\gamma_2,\gamma_3)} = x_{\overline{\gamma}_1}^{-1} x_{\gamma_2} x_{\overline{\gamma}_3}^{-1}$, the second summation is over all clockwise loops ℓ_p enclose an orbifold point p with $s(\ell_p) = i$.

Total angle

Proposition

Let Δ, Δ' be two triangulations of Σ. We have

$$T_i(\Delta) = T_i(\Delta').$$

We refer it as the *total angle at* i.

An equivalent presentation of $\mathcal{A}_\Sigma$:

- Angle at a vertex of any triangle is well-defined.
- Angles at any marked point are additive.

Noncommutative surfaces – Tagged cluster variables

i puncture, γ is a curve with $t(\gamma) = i$, define $x_{\gamma^{(i)}} = x_\gamma T_i$.

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i, j punctures, γ is a curve with $s(\gamma) = i, t(\gamma) = j$, define $x_{\gamma^{(i,j)}} = T_i x_\gamma T_j$.

Call them *tagged (non-commutative) cluster variables*.

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Theorem

For any puncture i the assignments $x_\gamma \mapsto T_i^{\delta_{i,s(\gamma)}} x_\gamma T_i^{\delta_{i,t(\gamma)}}$ define an involutive automorphism φ_i of $\mathcal{A}_\Sigma$, where $s(\gamma)$ and $t(\gamma)$ are respectively the starting and terminating point of γ.
Moreover, these automorphisms commute so that for any subset P of punctures the composition φ_P of all φ_i, $i \in P$ is well-defined.

Triangle group and braid group action

Δ: ideal triangulation of Σ, the *triangle group* $\mathbb{T}_\Delta = \langle t_\gamma, \gamma \in \Delta \rangle$ subject to

$$t_{\gamma_1} t_{\bar{\gamma}_2}^{-1} t_{\gamma_3} = t_{\bar{\gamma}_3} t_{\gamma_2}^{-1} t_{\bar{\gamma}_1}$$

for any triangle $(\gamma_1, \gamma_2, \gamma_3)$ in Δ.

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Δ tagged triangulation of Σ, the *triangle group* $\mathbb{T}_\Delta = \langle t_\gamma, \gamma \in \Delta \rangle$ subject to

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- $t_{10} t_{01^{tag}} = t_{10^{tag}} t_{01}$;
- $t_{21}^{+} (t_{10} t_{01^{tag}})^{-1} t_{12}^{-} = t_{21}^{-} (t_{10}^{tag} t_{01})^{-1} t_{12}^{+}$ for any once-punctured digon.

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Theorem (Noncommutative Laurent Phenomenon)

The extension $\iota_\Delta : \mathbb{Q}\mathbb{T}_\Delta \to \mathcal{A}_\Sigma$ is injective for any tagged triangulation Δ of Σ and all x_γ (γ is a tagged curve) belong to its image. More precisely, each x_γ can be uniquely expressed as a positive sum of elements of $\iota_\Delta(\mathbb{T}_\Delta)$.

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In fact, the total angles T_i are in the image of all ι_Δ.

Triangle group and braid group action

Fix an ideal triangulation Δ of Σ, if γ is not a self-folded arc, define $T_\gamma \in Aut(\mathbb{T}_\Delta)$

$$T_\gamma(t_\alpha) = \begin{cases} t_{ij}t_{kj}^{-1}t_{kl}t_{il}^{-1}t_\gamma & \text{if } \gamma = (ik) \\ t_\gamma t_{li}^{-1}t_{lk}t_{jk}^{-1}t_{ji} & \text{if } \gamma = (ki) \\ t_\gamma & \text{otherwise} \end{cases}$$

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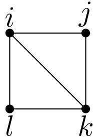

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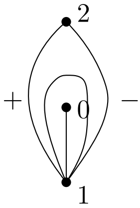

If $\gamma = (0,1)$ is self-folded arc, define $T_\gamma \in Aut(\mathbb{T}_\Delta)$

$$T_\gamma(t_\alpha) = \begin{cases} t_{12}^{+}(t_{12}^{-})^{-1}t_{10} & \text{if } \gamma = (10) \\ \bar{T}_\gamma(10) & \text{if } \gamma = (01) \\ t_{12}^{+}(t_{12}^{-})^{-1}t_{11} & \text{if } \gamma = (11) \\ t_\gamma & \text{otherwise} \end{cases}$$

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Theorem

For any two triangulations Δ, Δ' of Σ, $Br_\Delta \cong Br_{\Delta'}$, moreover,

(a) *if α and β are not two sides of any triangle in Δ, then $T_\alpha T_\beta = T_\beta T_\alpha$;*

(b) *If α and β are two sides of exactly one triangle in Δ, then $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$;*

(c) *If (α_1, α_2) forms a once-punctured bigon with diagonals α_3 and α_4, assume that $(\alpha_1, \alpha_3, \alpha_4)$ is a clockwise triangle in Δ and α_1, α_2 are not two sides of any triangle in Δ, then $T_{\alpha_1} T_{\alpha_4} T_{\alpha_2} T_{\alpha_3} T_{\alpha_1} T_{\alpha_4} = T_{\alpha_4} T_{\alpha_2} T_{\alpha_3} T_{\alpha_1} T_{\alpha_4} T_{\alpha_2}$;*

Continue

Theorem

For any two triangulations Δ, Δ' *of* Σ, $Br_\Delta \cong Br_{\Delta'}$, *moreover,*

(d) *If* $(l(\alpha), \alpha_1, \alpha_2)$ *and* $(l(\beta), \alpha_1, \alpha_2)$ *are two different clockwise cyclic triangles in* Δ *such that* $l(\alpha)$ *and* $l(\beta)$ *are not two sides of any triangle in* Δ, *then*

$$T_\alpha T_{\alpha_1} T_{\alpha_2} T_\alpha T_{\alpha_1} T_{\alpha_2} = T_{\alpha_1} T_{\alpha_2} T_\alpha T_{\alpha_1} T_{\alpha_2} T_\alpha,$$
$$T_\beta T_{\alpha_1} T_{\alpha_2} T_\beta T_{\alpha_1} T_{\alpha_2} = T_{\alpha_1} T_{\alpha_2} T_\beta T_{\alpha_1} T_{\alpha_2} T_\beta;$$

(e) *If* $\alpha_1, \alpha_2, \ldots, \alpha_n$ $(n \geq 2)$ *are all the arcs incident to some puncture* $p \in I_p$, *assume that* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are in clockwise order and* $\alpha_i \neq \alpha_j$ *for any* $i \neq j$, *then*

$$T_{\alpha_1} T_{\alpha_2} \ldots T_{\alpha_n} T_{\alpha_1} T_{\alpha_2} \ldots T_{\alpha_{n-2}} = T_{\alpha_2} T_{\alpha_3} \ldots T_{\alpha_n} T_{\alpha_1} T_{\alpha_2} \ldots T_{\alpha_{n-1}}.$$

Triangle group and braid group action

Remark

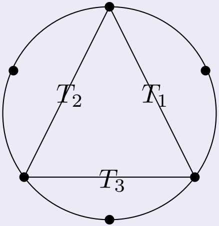
1. The groups satisfy relations (a)-(c), (e) appeared in terms of quiver with potential by Qiu, Qiu-Zhou, King-Qiu, etc;
2. The automorphisms T_γ is inspired by the symplectic transvection defined by Shapiro-Shapiro-Vainshtein-Zelevinsky.

Center of the braid groups Br_A, Br_D

Theorem[Berenstein-H-Retakh] Let (S, M) be an n-gon or a once-punctured n-gon and T be a triangulation of (S, M). Assume that $\lhd$ is a linear order on T which is compatible with $Q(T)$. Then $(\prod_{i\in T}^{\lhd} \tau_i)^c$ is the generator of the center of Br_T, where c is the Coxeter number.

Symmetries for surfaces

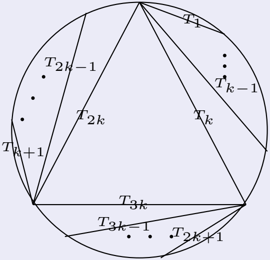
Example Rotation $2\pi/3$ gives an automorphism of the hexagon, it induces an automorphism of $\mathbb{T}_T$ and an automorphism φ of Br_4.



Then φ is inner and $\varphi(-) = \tau \cdot - \cdot \tau^{-1}$, where $\tau = T_1 T_2 T_3 T_1$ and $\tau^3 \in Z(Br_4) = \langle \tau^3 \rangle$.

Symmetries for surfaces

Example Rotation $2\pi/3$ gives an automorphism of the hexagon, it induces an automorphism φ of Br_{3k+1}. Denote $\tau_i = T_i T_{k+i} T_{2k+i}$ for $1 \leq i \leq k-1$ and $\tau_k = T_k T_{2k} T_{3k} T_k$.



Then φ is inner and $\varphi(-) = \tau \cdot - \cdot \tau^{-1}$, where $\tau = (\tau_1 \cdots \tau_{k-1}\tau_k)^k$ and $\tau^3 \in Z(Br_{3k+1}) = \langle \tau^3 \rangle$.

谢谢!